Decoherence in quantum walks on the hypercube

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We study a natural notion of decoherence on quantum random walks over the hypercube. We prove that this model possesses a decoherence threshold beneath which the essential properties of the hypercubic quantum walk, such as linear mixing times, are preserved. Beyond the threshold, we prove that the walks behave like their classical counterparts.

I. INTRODUCTION

The notion of a *quantum random walk* has emerged as an important element in the development of efficient quantum algorithms. In particular, it makes a dramatic appearance in the most efficient known algorithm for element distinctness [1]. The technique has also provided simple separations between quantum and classical query complexity [2], improvements in mixing times over classical walks [3, 4], and some interesting search algorithms [5, 6].

The basic model has two natural variants, the *continuous* model of Childs, et al. [7], on which we will focus, and the *discrete* model introduced by Aharonov, et al. [8]. We refer the reader to Szegedy's [9] article for a more detailed discussion. In the continuous model, a quantum walk on a graph G is determined by the time-evolution of the Schrödinger equation using kL as the Hamiltonian, where L is the Laplacian of the graph and k is a positive scalar to which we refer as the "jumping rate" or "energy". In addition to being a physically attractive model, it has been successfully applied to some algorithmic problems as indicated above.

Such walks have been studied over a variety of graphs with special attention given to Cayley graphs, whose algebraic structure has provided immediate methods for determining the spectral resolution of the linear operators that determine the system's dynamics. Once it had been discovered that quantum random walks can offer improvement over their classical counterparts with respect to such basic phenomena as mixing and hitting times, it was natural to ask how robust these walks are in the face of decoherence, as this would presumably be an issue of primary importance for any attempt at implementation [10, 11, 12].

In this article, we study the effects of a natural notion of decoherence on the hypercubic quantum walk. Our notion of decoherence corresponds, roughly, to independent measurement "accidentally" taking place in each coordinate of the walk at a certain rate p. We discover that for values of p beneath a threshold depending on the energy of the system, the walk retains the basic features of the non-decohering walk; these features disappear beyond this threshold, where the behavior of the classical walk is recovered.

Moore and Russell [4] analyzed both the discrete and the continuous quantum walk on a hypercube. Kendon and Tregenna [13] performed a numerical analysis of the effect of decoherence in the discrete case. In this article, we extend the continuous case with the model of decoherence described above. In particular, we show that up to a certain rate of decoherence, both linear instantaneous mixing times and linear instantaneous hitting times still occur. Beyond the threshold, however, the walk behaves like the classical walk on the hypercube, exhibiting $\Theta(n \log n)$ mixing times. As the rate of decoherence grows, mixing is retarded by the quantum Zeno effect.

A. Results

Consider the continuous quantum walk on the n-dimensional hypercube with energy k and decoherence rate p, starting from the initial wave function $\Psi_0 = |0\rangle^{\otimes n}$, corresponding to the corner with Hamming weight zero. We prove the following theorems about this walk.

Theorem 1. *When $p < 4k$, the walk has instantaneous mixing times at*

$$t_{mix} = \frac{n(2\pi c - \arccos(p^2/8k^2 - 1))}{\sqrt{16k^2 - p^2}}$$

for all $c \in \mathbb{Z}$, $c > 0$. At these times, the total variation distance

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between the walk distribution and the uniform distribution is zero.

This result is an extension of the results in [4], and an improvement over the classical random walk mixing time of $\Theta(n \log n)$. Note that the mixing times decay with p and disappear altogether when $p \geq 4k$. Further, for large p, we will see that the walk is retarded by the quantum Zeno effect.

Theorem 2. *When $p < 4k$, the walk has approximate instantaneous hitting times to the opposite corner $(1, \ldots, 1)$ at times*

$$t_{hit} = \frac{2\pi n(2c+1)}{\sqrt{16k^2 - p^2}}$$

for all $c \in \mathbb{Z}$, $c \geq 0$. However, the probability of measuring an exact hit decays exponentially in c; the probability is

$$P_{hit} = \left[\frac{1}{2} + \frac{1}{2} e^{-\frac{p\pi(2c+1)}{\sqrt{16k^2 - p^2}}} \right]^n .$$

In particular, when no decoherence is present, the walk hits at $t_{hit} = \frac{n\pi(2c+1)}{2k}$, and it does so exactly, i.e. $P_{hit} = 1$. For $p \geq 4k$, no such hitting occurs.

This result is a significant improvement over the exponential hitting times of the classical random walk, with the caveat that decoherence has a detrimental effect on the accuracy of repeated hitting times.

Finally, we show that under high levels of decoherence, the measurement distribution of the walk actually converges to the uniform distribution in time $\Theta(n \log n)$, just as in the classical case.

Theorem 3. *For a fixed $p \geq 4k$, the walk mixes in time $\Theta(n \log n)$.*

In the remainder of the introduction, we describe the continuous quantum walk model, and recall the graph product analysis of Moore and Russell [4]. In the second section, we describe our model of decoherence, derive a superoperator that governs the behavior of the decohering walk, and prove that it is decomposable into an n-fold tensor product of a small system. We then fully analyze the small system in the third section, and use those results to draw conclusions about the general walk in 3 distinct regimes: $p < 4k$, $p = 4k$, and $p > 4k$. These regimes are roughly analogous to underdamping, critical damping, and overdamping (respectively) of a simple harmonic oscillator with damping rate p and angular frequency $2k$.

B. The continuous quantum walk on the hypercube

A continuous quantum walk on a graph G begins at a distinguished vertex v_0 of G, the initial wave function of the walk being Ψ_0, where $\langle \Psi_0 | v \rangle = 1$ if $v = v_0$ and 0 otherwise. The walk then evolves according to the Schrödinger equation. In our case, the graph is the n-dimensional hypercube. Concretely, we identify the vertices with n-bit strings, with edges connecting those pairs of vertices that differ in exactly one bit. Since the hypercube is a regular graph, we can let the Hamiltonian H be the adjacency matrix instead of the Laplacian [14]; the dynamics are then given by the unitary operator $U_t = e^{iHt}$ and the state of the walk at time t is $\Psi_t = U_t \Psi_0$.

The following analysis makes use of the hypercube's product graph structure; this structure will be useful again later when we consider the effects of decoherence. The analysis below diverges from that of Moore and Russell [4] only in that we allow each qubit to have energy k/n instead of $1/n$. The energy of the entire system is then k. Let

$$\sigma_x = \begin{pmatrix} 0 & k/n \\ k/n & 0 \end{pmatrix},$$

and let

$$H = \sum_{j=1}^{n} \mathbb{1} \otimes \cdots \otimes \sigma_x \otimes \cdots \otimes \mathbb{1} ,$$

where the jth term in the sum has σ_x as the jth factor in the tensor product. Then we have

$$U_t = e^{iHt} = \prod_{j=1}^{n} \mathbb{1} \otimes \cdots \otimes e^{it\sigma_x} \otimes \cdots \otimes \mathbb{1} = \left[e^{it\sigma_x} \right]^{\otimes n}$$

$$= \begin{bmatrix} \cos(kt/n) & i \sin(kt/n) \\ i \sin(kt/n) & \cos(kt/n) \end{bmatrix}^{\otimes n} .$$

Applying U_t to the initial state $\Psi_0 = |0\rangle^{\otimes n}$, we have

$$U_t \Psi_0 = \left[\cos\left(\frac{kt}{n}\right) |0\rangle + i \sin\left(\frac{kt}{n}\right) |1\rangle \right]^{\otimes n}$$

which corresponds to a uniform state exactly when $\frac{kt}{n}$ is an odd multiple of $\frac{\pi}{4}$.

II. A DERIVATION OF THE SUPEROPERATOR

We begin by recalling a model of decoherence commonly used in the discrete model, with the intention of deriving a superoperator U_t, acting on density matrices, which mimics these dynamics in our continuous setting. The discrete model, described in [13], couples unitary evolution according to the

discrete-time quantum random walk model of Aharonov et al. [8] with partial measurement at each step occurring with some fixed probability p. Specifically, the evolution of the density matrix can be written as

$$\rho_{t+1} = (1-p)U\rho_t U^\dagger + p\sum_i \mathbf{P_i} U\rho_t U^\dagger \mathbf{P_i}$$

where U is the unitary operator of the walk, i runs over the dimensions where the decoherence occurs, and the $\mathbf{P_i}$ project in the usual "computational" basis [13].

In the continuous setting, the unitary operator that governs the non-decohering walk is $U_t = e^{-iHt}$, where H is the normalized adjacency matrix of the hypercube times an energy constant. To extend the above decoherence model to this setting, recall that the superoperator $U_t \otimes U_t^\dagger$ associated with these dynamics has the property that

$$\frac{dU_t \otimes U_t^\dagger}{dt} = i\left(e^{-iHt} \otimes e^{iHt}\right)[\mathbb{1} \otimes H - H \otimes \mathbb{1}] \;;$$

wishing to augment these dynamics with measurement occurring at some prescribed rate p, we desire a superoperator S_t that satisfies

$$S_{t+dt} = S_t[e^{-iHdt} \otimes e^{iHdt}][(1-p\,dt)\mathbb{1} + pdt(\mathbf{P})]$$

where $\mathbf{P}$ is the operator associated with the decohering measurement. Intuitively, the unitary evolution of the system is punctuated by measurements taking place with rate p, analogous to the discrete case.

Letting $e^{-iHdt} = \mathbb{1} - iHdt$, we can expand and simplify:

$$
\begin{aligned}
S_{t+dt} &= S_t[e^{-iHdt} \otimes e^{iHdt}][(1-pdt)\mathbb{1} + pdt(\mathbf{P})] \\
&= S_t[(\mathbb{1} - iHdt) \otimes (\mathbb{1} + iHdt)][(1-pdt)\mathbb{1} + pdt(\mathbf{P})] \\
&= S_t[\mathbb{1} \otimes \mathbb{1} + idt(\mathbb{1} \otimes H - H \otimes \mathbb{1}) - pdt\,\mathbb{1} \otimes \mathbb{1} + pdt(\mathbf{P})]\;.
\end{aligned}
$$

In terms of a differential equation,

$$
\begin{aligned}
\frac{dS_t}{dt} &= \frac{S_{t+dt} - S_t}{dt} \\
&= \frac{S_t[\mathbb{1} \otimes \mathbb{1} + idt(\mathbb{1} \otimes H - H \otimes \mathbb{1}) - pdt\,(\mathbb{1} \otimes \mathbb{1} + \mathbf{P})] - S_t}{dt} \\
&= S_t[i(\mathbb{1} \otimes H - H \otimes \mathbb{1}) - p\mathbb{1} \otimes \mathbb{1} + p(\mathbf{P})]\;.
\end{aligned}
$$

The solution is

$$S_t = \exp\left([i(\mathbb{1} \otimes H - H \otimes \mathbb{1}) - p\mathbb{1} \otimes \mathbb{1} + p(\mathbf{P})]t\right)\;. \quad \text{(II.1)}$$

We now define the decoherence operator $\mathbf{P}$. This operator will correspond to choosing a coordinate uniformly at random and measuring it by projecting to the computational basis $\{|0\rangle, |1\rangle\}$. Let Π_0 and Π_1 be the single qubit projectors onto $|0\rangle$ and $|1\rangle$, respectively. We define

$$\mathbf{P} = \frac{1}{n}\sum_{1 \le i \le n}[\Pi_0^i \otimes \Pi_0^i + \Pi_1^i \otimes \Pi_1^i]$$

where $\Pi_0^i = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \Pi_0 \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$ with the nonidentity projector appearing in the ith place. We define Π_1^i similarly, so that Π_j^i ignores all the qubits except the ith one, and projects it onto $|j\rangle$ where $j \in \{0, 1\}$. Note that

$$\Pi_j^i \otimes \Pi_j^i = [\mathbb{1} \otimes \mathbb{1}] \otimes \cdots \otimes [\Pi_j \otimes \Pi_j] \otimes \cdots \otimes [\mathbb{1} \otimes \mathbb{1}]$$

for $j \in \{0, 1\}$.

A. The superoperator as an n-fold tensor product

The pure continuous quantum walk on the n-dimensional hypercube is easy to analyze, in part, because it is equivalent to a system of n non-interacting qubits. We now show that, with the model of decoherence described above, each dimension still behaves independently. In particular, the superoperator that dictates the behavior of the walk is decomposable into an n-fold tensor product.

Recall the product formulation of the non-decohering Hamiltonian

$$H = \sum_{j=1}^n \mathbb{1} \otimes \cdots \otimes \sigma_x \otimes \cdots \otimes \mathbb{1}$$

where

$$\sigma_x = \begin{pmatrix} 0 & k/n \\ k/n & 0 \end{pmatrix}$$

with σ_x appearing in the jth place in the tensor product. We have given each single qubit energy k/n, resulting in a system with energy k. This choice will allow us to precisely describe the behavior of the walk in terms of the relationship between the energy of the system and the rate of decoherence.

We can write each of the terms in the exponent of the superoperator from (II.1) as follows:

$$\mathbb{1} \otimes H = \sum_{j=1}^n [\mathbb{1} \otimes \mathbb{1}] \otimes \cdots \otimes [\mathbb{1} \otimes \sigma_x] \otimes \cdots \otimes [\mathbb{1} \otimes \mathbb{1}]\;,$$

$$H \otimes \mathbb{1} = \sum_{j=1}^n [\mathbb{1} \otimes \mathbb{1}] \otimes \cdots \otimes [\sigma_x \otimes \mathbb{1}] \otimes \cdots \otimes [\mathbb{1} \otimes \mathbb{1}]\;.$$

Our decoherence operator can also be written in this form:

$$
\begin{aligned}
\mathbf{P} &= \frac{1}{n}\sum_{j=1}^n [\Pi_0^i \otimes \Pi_0^i + \Pi_1^i \otimes \Pi_1^i] \\
&= \frac{1}{n}\sum_{j=1}^n ([\mathbb{1} \otimes \mathbb{1}] \otimes \cdots \otimes [\Pi_0 \otimes \Pi_0] \otimes \cdots \otimes [\mathbb{1} \otimes \mathbb{1}] \\
&\quad + [\mathbb{1} \otimes \mathbb{1}] \otimes \cdots \otimes [\Pi_1 \otimes \Pi_1] \otimes \cdots \otimes [\mathbb{1} \otimes \mathbb{1}])\;.
\end{aligned}
$$

The identity operator has a consistent decomposition: $\mathbb{1} \otimes \mathbb{1} = \frac{1}{n}\sum_{j=1}^n [\mathbb{1} \otimes \mathbb{1}] \otimes \cdots \otimes [\mathbb{1} \otimes \mathbb{1}]$. We can now put these pieces

together to form the superoperator:

$$S_t = \exp\left(it(\mathbb{1} \otimes H) - it(H \otimes \mathbb{1}) - pt\,\mathbb{1} \otimes \mathbb{1} + pt\mathbf{P}\right)$$

$$= \exp\left(\sum_{j=1}^{n} [\mathbb{1} \otimes \mathbb{1}] \otimes \cdots \otimes \mathbf{A} \otimes \cdots [\mathbb{1} \otimes \mathbb{1}]\right)$$

$$= \prod_{j=1}^{n} [\mathbb{1} \otimes \mathbb{1}] \otimes \cdots \otimes e^{\mathbf{A}} \otimes \cdots [\mathbb{1} \otimes \mathbb{1}]$$

$$= \left[e^{\mathbf{A}}\right]^{\otimes n}$$

where

$$
\begin{aligned}
\mathbf{A} &= \frac{t}{n}[(\mathbb{1} \otimes in\sigma_x) - (in\sigma_x \otimes \mathbb{1}) - p(\mathbb{1} \otimes \mathbb{1}) \\
&\quad + p(\Pi_1 \otimes \Pi_1) + p(\Pi_0 \otimes \Pi_0)] \\
&= \frac{t}{n}\begin{pmatrix} 0 & ik & -ik & 0 \\ ik & -p & 0 & -ik \\ -ik & 0 & -p & ik \\ 0 & -ik & ik & 0 \end{pmatrix}.
\end{aligned}
$$

Notice that for $p = 0$, $[e^{\mathbf{A}}]^{\otimes n} = [e^{-it\sigma_x} \otimes e^{it\sigma_x}]^{\otimes n}$, which is exactly the superoperator formulation of the dynamics of the non-decohering walk.

III. SMALL-SYSTEM BEHAVIOR AND ANALYSIS OF THE WALK

So far we have shown that the walk with decoherence is still equivalent to n non-interacting single-qubit systems. We now analyze the behavior of a single-qubit system under the superoperator $e^{\mathbf{A}}$. The structure of this single particle walk will allow us to then immediately draw conclusions about the entire system.

The eigenvalues of $\mathbf{A}$ are 0, $-\frac{pt}{n}$, $\frac{-pt-\alpha t}{2n}$ and $\frac{-pt+\alpha t}{2n}$. Here $\alpha = \sqrt{p^2 - 16k^2}$ is a complex constant that will later turn out to be important in determining the behavior of the system as a function of the rate of decoherence p and the energy k. The matrix exponential of $\mathbf{A}$ in this spectral basis can then be computed by inspection. To see how our superoperator acts on a density matrix ρ_0, we may change ρ_0 to the spectral basis, apply the diagonal superoperator to yield ρ_t, and finally change ρ_t back to the computational basis. At that point we can apply the usual projectors Π_0 and Π_1 to determine the probabilities of measuring 0 or 1 in terms of time.

Let $\Psi_0 = |0\rangle$ and $\rho_0 = |\Psi_0\rangle\langle\Psi_0|$. In the diagonal basis,

$$
\rho_0 = \begin{bmatrix} 1/2 \\ 0 \\ \frac{1}{4}(-1 + \frac{p}{\alpha}) \\ \frac{1}{4}(-1 - \frac{p}{\alpha}) \end{bmatrix}
$$

and thus at time t we have

$$
\rho_t = e^{\mathbf{A}}\rho_0 = \begin{bmatrix} 1/2 \\ 0 \\ \frac{1}{4}e^{\frac{-pt-\alpha t}{2n}}(-1 + \frac{p}{\alpha}) \\ \frac{1}{4}e^{\frac{-pt+\alpha t}{2n}}(-1 - \frac{p}{\alpha}) \end{bmatrix}.
$$

If we then change back to the computational basis and project by Π_0 and Π_1, we may compute the probabilities of measuring 0 and 1 at a particular time t:

$$P[0] = \frac{1}{4}\left[2 + e^{\frac{-pt-\alpha t}{2n}}(1 - p/\alpha) + e^{\frac{-pt+\alpha t}{2n}}(1 + p/\alpha)\right]$$

$$P[1] = \frac{1}{4}\left[2 - e^{\frac{-pt-\alpha t}{2n}}(1 - p/\alpha) - e^{\frac{-pt+\alpha t}{2n}}(1 + p/\alpha)\right]$$

which can be simplified somewhat to

$$P[0] = \frac{1}{2} + \frac{1}{2}e^{-\frac{pt}{2n}}\left[\cos\left(\frac{\beta t}{2n}\right) + \frac{p}{\beta}\sin\left(\frac{\beta t}{2n}\right)\right]$$

$$P[1] = \frac{1}{2} - \frac{1}{2}e^{-\frac{pt}{2n}}\left[\cos\left(\frac{\beta t}{2n}\right) + \frac{p}{\beta}\sin\left(\frac{\beta t}{2n}\right)\right].$$

Here we have let $\beta = -i\alpha = \sqrt{16k^2 - p^2}$ for simplicity. A quick check shows that when $p = 0$, $P[0] = \cos^2\left(\frac{kt}{n}\right)$ and $P[1] = \sin^2\left(\frac{kt}{n}\right)$, which are exactly the dynamics of the non-decohering walk. The probabilities for this non-decohering case are shown in Figure 1.
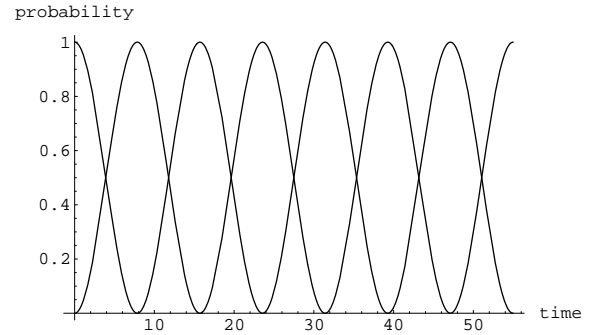

FIG. 1: The $p = 0$ case - no decoherence: a plot of $P[0]$ and $P[1]$ versus time, for $k = 1, n = 5, p = 0$

The three regimes mentioned before are immediately apparent. For $p < 4k$, β is real. When $p = 4k$, we have $\beta = 0$, which appears to be a serious problem at first glance. Finally, for $p > 4k$, β is imaginary. We now address each of these three situations in detail.

A. The case $p < 4k$: linear mixing and hitting times

When $p < 4k$, we recover the perhaps most interesting feature of the non-decohering walk: the instantaneous mixing time is linear in n. To exactly determine the mixing times for our decohering walk, we solve $P[0] = P[1] = \frac{1}{2}$; this amounts to determining when

$$\gamma = \frac{1}{2}e^{-\frac{pt}{2n}}\left[\cos\left(\frac{\beta t}{2n}\right) + \frac{p}{\beta}\,\sin\left(\frac{\beta t}{2n}\right)\right]$$

equals zero. Clearly the exponential decay term results in mixing as $t \to \infty$; our principle concern, however, is with the periodic mixing times analogous to those of the original walk. We thus ignore the exponential term when solving the equality $\gamma = 0$, which yields

$$\frac{p^2}{\beta^2} = \frac{1 + \cos(\beta t/n)}{1 - \cos(\beta t/n)}.$$

This equation actually has more solutions than the one we started with, because of the use of half-angle formulas for simplification. The solutions that we want are

$$t_{mix} = \frac{n}{\beta}\left[2\pi c - \arccos\left(\frac{p^2}{8k^2} - 1\right)\right]$$

where c ranges over the positive integers. Evidently, the mixing times still occur in linear time; an example is shown in Figure 2. Note also that if we let $p = 0$, we have $t_{mix} = n\pi(2c-1)/(4k)$, which are exactly the nice periodic mixing times of the non-decohering walk. In the decohering case, however, these mixing times drift towards infinity, and cease to exist altogether beyond the threshold of $p = 4k$. This proves Theorem 1.

We now wish to determine when our small system is as close as possible to $|1\rangle$. Since our large-system walk begins at $|0\rangle^{\otim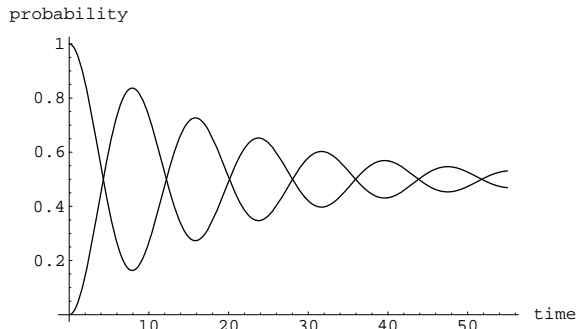es n}$, this will correspond to approximate hitting times to the opposite corner $|1\rangle^{\otimes n}$. These times correspond to local maxima of $P[1]$; the solutions are

$$t_{hit} = 2n\pi\left(\frac{2c+1}{\beta}\right)$$

where c ranges over the non-negative integers. At these points in time, the value of $P[1]$ is

$$\frac{1}{2} + \frac{1}{2}e^{-(2c+1)\frac{p\pi}{\beta}}$$

which immediately yields Theorem 2.

B. The breakpoint case $p = 4k$

We first observe that $t_{mix} \to \infty$ as $p \to 4k$. Hence, we do not expect to see any mixing in this case. To analyze the probabilities exactly, we take the limit of γ as $p \to 4k$. The solution is

$$\lim_{p \to 4k} \gamma = \frac{1}{2}e^{-\frac{2kt}{n}}\left[1 + \frac{2kt}{n}\right]\ . \tag{III.1}$$

Indeed, since k, t and n are all positive, γ is zero only in the limit as $t \to \infty$. The linear mixing and hitting behavior from the previous section has entirely disappeared. As in the critical damping of simple harmonic motion, a small decrease in the rate p can result in drastically different behavior, in this case a return to linear mixing and hitting. We leave the limiting mixing analysis of this case for the next section, where we develop some relevant tools.

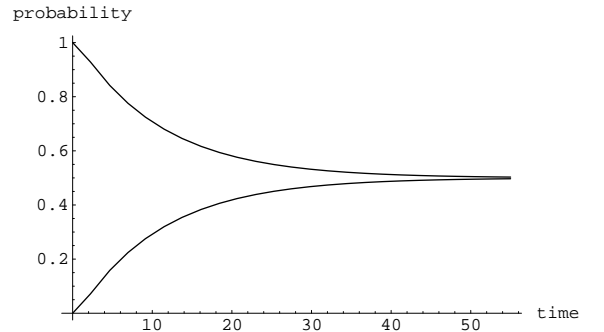## C. The case $p > 4k$ and the limit to the classical walk

FIG. 3: The $p > 4k$ case: a plot of $P[0]$ and $P[1]$ versus time, for $k = 1$, $n = 5$, $p = 9$

The goal of this section is to show two interesting consequences of the presence of substantial decoherence in the

FIG. 2: The $p < 4k$ case: a plot of $P[0]$ and $P[1]$ versus time, for $k = 1$, $n = 5$, $p = 0.5$

quantum walk on the hypercube. First, we will show that for a fixed $p \geq 4k$, the walk behaves much like the classical walk on the hypercube, mixing in time $\Theta(n \log n)$. Second, we show that as $p \to \infty$, the walk suffers from the quantum Zeno effect. Informally stated, the rate of decoherence is so large that the walk is continuously being reset to the initial wave function $|0\rangle^{\otimes n}$ by measurement.

1. Recovering classical behavior

Consider a single qubit. Let P be the distribution obtained by full measurement at time t, and U the uniform distribution:

$$P(0) = \frac{1}{2} + \gamma, \qquad P(1) = \frac{1}{2} - \gamma, \qquad U(0) = U(1) = \frac{1}{2},$$

where

$$\gamma = \frac{1}{4}\left[e^{\frac{(-p-\alpha)t}{2n}}(1 - p/\alpha) + e^{\frac{(\alpha - p)t}{2n}}(1 + p/\alpha) \right].$$

For $x = (x_1, \ldots, x_n) \in \mathbb{Z}_2^n$,

$$P^n(x) = \prod_{i=1}^{n} P(x_i) \qquad \text{and} \qquad U^n(x) = 2^{-n}$$

are the analogous product distributions in the n-dimensional case. To analyze the limiting mixing behavior of the walk, we will consider the total variation distance $\|P^n - U^n\| = \sum_x |P^n(x) - U^n(x)|$ between these distributions. In order to give bounds for total variation, we will use *Hellinger distance* [15], defined as follows:

$$H(A,B)^2 = \sum_x \left(\sqrt{A(x)} - \sqrt{B(x)} \right)^2 = 1 - \sum_x \sqrt{A(x)B(x)}.$$

We will make use of the following two properties of Hellinger distance:

$$1 - H(A^n, B^n)^2 = (1 - H(A,B)^2)^n \ ,$$

and

$$\|A - B\| \leq 2H(A,B) \leq 2\|A - B\|^{1/2}. \qquad \text{(III.2)}$$

The first property makes it easy to work with product distributions. The second gives a nice relationship between Hellinger distance and total variation distance. In our case,

$$\begin{aligned} H(P^n, U^n)^2 &= 1 - (1 - H(P,U)^2)^n \\ &= 1 - \left(\frac{1}{2}\sqrt{1+2\gamma} + \frac{1}{2}\sqrt{1-2\gamma} \right)^n \\ &= 1 - \left(1 - \frac{\gamma^2}{2} + O(\gamma^3) \right)^n. \end{aligned}$$

And hence, by (III.2),

$$\|P_n - U_n\|^2 \leq 4 - 4\left(1 - \frac{\gamma^2}{2} + O(\gamma^3) \right)^n.$$

Consider the walk with decoherence rate $p > 4k$. We have $\alpha = \sqrt{p^2 - 16k^2} < p$, where α and p are positive and real. It follows that for a fixed $p > 4k$, $\gamma \to 0$ and $\|P^n - U^n\| \to 0$ as $t \to \infty$. Hence the walk does indeed mix eventually, and the measurement distribution in fact converges to the uniform distribution. Let $t = d \cdot n \log n$ where $d > 0$ is a constant, and rewrite γ as follows:

$$\gamma = \frac{1}{4} e^{-(p-\alpha)\frac{d \log n}{2}} \left[(1 - p/\alpha) + e^{\frac{-\alpha d \log n}{2}}(1 + p/\alpha) \right].$$

Suppose we choose d such that $d > (p - \alpha)^{-1}$. Then $\gamma = o(n^{-1/2})$, which implies that $\|P^n - U^n\| = o(1)$. On the other hand, if $d < (p - \alpha)^{-1}$, then $\gamma = \omega(n^{-1/2})$ and there exists a constant ε such that $\|P^n - U^n\| \geq \varepsilon > 0$. This shows that the walk mixes in time $\Theta(n \log n)$ when $p > 4k$. Notice that when $p = 4k$, $(p - \alpha)^{-1} = (4k)^{-1}$, so that the same technique easily extends to that case via equation (III.1). This completes the proof of Theorem 3.

2. Quantum Zeno effect for large p

Recall from the previous section that the time required to mix when $p > 4k$ is

$$t \geq \frac{n \log n}{p - \alpha}$$

which clearly increases with p. Further, for large p, p/α tends to 1, and hence γ tends to $1/2$. Notice that $\gamma = 1/2$ corresponds to remaining at the initial state forever. We conclude that the mixing of the walk is retarded by the quantum Zeno effect, where measurement occurs so often that the system tends to remain in the initial state.

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